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Arithmeticity of rank-1 lattices with dense commensurators in positive characteristic

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Abstract

G. Margulis showed that if G is a semisimple Lie group and $\Gamma \subset G$ is an irreducible lattice, which has an infinite index in its commensurator, and which satisfies one of the following conditions: (1) it is cocompact; (2) at least one of the simple components of G is defined over a local field of characteristic 0; (3) $\mathit{rank}\, G \geqslant 2$, then Γ is arithmetic. This leaves out the case of non-uniform lattices in rank-1 simple groups G defined over a local field of positive characteristic. We show the arithmeticity of the lattice Γ in this remaining case (under the assumption of density of its commensurator).

1. Introduction

G. Margulis studied the question of arithmeticity of lattices in semisimple Lie groups. He showed [4, Theorem IX.1.13] that if G is a semisimple Lie group and $\Gamma \subset G$ is an irreducible lattice, which has an infinite index in its commensurator $\mathit{Comm}_G\, \Gamma$, and which satisfies one of the following conditions: (1) it is cocompact; (2) at least one of the simple components of G is defined over a local field of characteristic 0; (3) $\mathit{rank}\, G \geqslant 2$; then Γ is arithmetic. This theorem leaves out the case of non-uniform lattices in rank-1 simple groups G defined over a local field of positive characteristic.

Lattices in semisimple Lie groups defined over local fields of positive characteristic were studied by T.N. Venkataramana [7], who generalized an earlier version of the arithmeticity theorem [3] to positive characteristic, and by M.S. Raghunathan [5], who studied the fundamental domains of irreducible lattices in positive characteristic. Whereas

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the above works resulted in showing arithmeticity of finitely generated lattices, such as, for example, irreducible lattices in a product of at least two rank-1 simple Lie groups defined over fields of positive characteristic, they didn't answer the arithmeticity question for non-uniform lattices in rank-1 simple Lie groups, since such lattices are not finitely generated (see [2]).

In [2] A. Lubotzky constructed infinite families of uniform and non-uniform non-arithmetic lattices in rank-1 simple groups defined over local fields of positive characteristic. He, however, did not assume these lattices to have an infinite index in their commensurators (the assumption made by G. Margulis in his theorem).

We prove the following theorem completing the remaining case in Margulis' Arithmeticity theorem:

Theorem 1. *Let* $\mathbf{G}$ *be an absolutely almost simple algebraic group defined over a local field* k *of positive characteristic with* $\mathrm{rank}_k\, G = 1$. *Let* $\Gamma \subset \mathbf{G}(k)$ *be a non-uniform lattice that has an infinite index in its commensurator* $\mathrm{Comm}_G\, \Gamma$. *Then* Γ *is arithmetic.*

We derive Theorem 1 from Theorem 2 below using methods of G. Margulis and T.N. Venkataramana.

Theorem 2. *Let* $\mathbf{G}$ *be an absolutely almost simple algebraic group defined over a local field* k *of positive characteristic with* $\mathrm{rank}_k\, G = 1$ *and* $\Gamma \subset \mathbf{G}(k)$ *be a non-uniform lattice of infinite index in* $\mathrm{Comm}_G\, \Gamma$. *Then there exists a subring* $\mathcal{A} \subset k$ *finitely generated over* $\mathbb{F}_p$, *such that* $\Gamma \subset GL_n(\mathcal{A})$.

The proof of Theorem 2 uses the description of the lattice Γ given in [2], which in its turn is based on the fundamental work of M.S. Raghunathan [5].

2. Preliminaries

In this section we will establish some notation and list some known results to be used throughout the paper. Let k be a local field of positive characteristic $p > 0$, then $k \cong \mathbb{F}_q((1/t))$, where $q = p^r$. Given a valuation $v : k \to \mathbb{R}^+$ on k, one can define the metric ρ on k by $\rho(x, y) = |x - y|_v$. We will define the metric on a vector space k^n also called ρ by $\rho((x_1, \dots, x_n), (y_1, \dots, y_n)) = \max_{1 \leqslant i \leqslant n} \rho(x_i, y_i)$. Let $\mathbf{G}$ be an algebraic group defined over k, then $G = \mathbf{G}(k)$ is a topological group. Fix a matrix representation of $\mathbf{G}$ and define $\|g\|$ for $g \in G \subset GL_n(k)$ by $\|g\| = \max_{i,j} |g_{ij}|_v$.

Let K be a global field of positive characteristic. Then it is a finite extension of a field of rational functions in one variable over a finite field. Let S be a set of valuations of the field K. An element $x \in K$ is called *S-integral* if $|x|_w \leqslant 1$ for each valuation $w \notin S$. The set of S-integral elements is a subring of the field K, which will be denoted by $K(S)$.

Let G be a locally compact group and μ_G be the right Haar measure on G. A discrete subgroup Γ of G is called a *lattice*, if $\mu_G(\Gamma \setminus G) < \infty$. A lattice Γ is said to be *cocompact* or *uniform*, if the quotient space $\Gamma \setminus G$ is compact, otherwise Γ is *non-uniform*. Let K be a global field and S be a set of valuations of K. Let $\mathbf{G}$ be a K-group. We call a subgroup of G

S-arithmetic, if it is commensurable with $\mathbf{G}(K(S))$. Note that if $\mathbf{G}$ is an absolutely almost simple group of k-rank 1, then a lattice $\Gamma \subset G(k)$ is arithmetic if it is commensurable to $\mathbf{G}(K(S))$, where S consists of only one valuation.

Consider the field k as an additive group $\mathbb{A}dd_k$. Let $\Lambda \subset \mathbb{A}dd_k$ be a lattice. If we identify k with $\mathbb{F}_q((1/t))$ and assume $v(t) = p$, then Λ must contain an element l_m, with $v(l_m) = p^m$ for each $m \geqslant M$, some $M \in \mathbb{N}$. Similarly if $\Lambda \subset k^n$ is a lattice then it must contain an element $L_{m,i} = (l_1, \ldots, l_i, \ldots, l_n)$ such that $v(l_i) = p^m$ for each $i \in \{1, 2, \ldots, n\}$ and all $m \geqslant M$. We will denote the smallest such M by $\Sigma(\Lambda)$. If $\mathbf{U}$ is a two step nilpotent algebraic group defined over k such that the center $\mathcal{Z}(U)$ and $U/\mathcal{Z}(U)$ are vector groups, and $\Delta \subset U$ is a lattice, then let $\Sigma(\Delta) = \max \Sigma(\mathcal{Z}(\Delta)), \Sigma(\Delta/\mathcal{Z}(\Delta))$.

Let $\mathbf{G}$ be an absolutely almost simple algebraic group defined over a local field k and of k-rank 1. Let $\mathbf{P} \subset \mathbf{G}$ be a minimal parabolic subgroup of $\mathbf{G}$. Then the unipotent radical $\mathbf{U}$ of $\mathbf{P}$ is at most 2-step nilpotent group such that its center $\mathcal{Z}(\mathbf{U})$ and $\mathbf{U}/\mathcal{Z}(\mathbf{U})$ are vector groups. Fix a maximal k-torus $\mathbf{T}$ in $\mathbf{P}$, then $\mathbf{T} = \mathbf{S} \cdot \mathbf{A}$, where $\mathbf{S} \subset \mathbf{T}$ is a maximal k-split torus and $\mathbf{A}$ is an anisotropic torus. The split torus $\mathbf{S}$ acts on $\mathcal{Z}(\mathbf{U})$ and $\mathbf{U}/\mathcal{Z}(\mathbf{U})$ by characters 2α and α, respectively.

Assume k is a local field of positive characteristic. We can associate to $G = \mathbf{G}(k)$ its Bruhat–Tits tree X (see [6]). Let us fix a vertex $x_0 \in X$. An infinite path with origin in x_0 and without backtracking is called a *ray* of X. Two rays are equivalent if their intersection is infinite. An equivalence class of rays is called an *end* of X. The set of ends ∂X is called the *boundary* of X. If $\epsilon \in \partial X$ is an end, then the stabilizer $G_\epsilon = \{g \in G \mid g\epsilon = \epsilon\}$ is a minimal parabolic subgroup P of G. We know $\mathbf{P} = \mathbf{S} \cdot \mathbf{A} \cdot \mathbf{U}$. If $S \cong k^*$, denote by S^1 the set of elements of S mapped onto the group of units of k^* by the above automorphism and $P^1 = S^1 \cdot A \cdot U$. If Γ is a lattice in G, then Theorem 4.1 in [5] says that $P^1/P^1 \cap \Gamma$ is compact.

Let Γ be a non-uniform lattice in G ($rank_k(G) = 1$). A. Lubotzky described the structure of all such lattices in the following theorem.

Theorem 3 [2, Theorem 7.1]. *Γ contains a finite index subgroup Γ' with the following structure: there exist ends $\epsilon_1, \ldots, \epsilon_c$ in X such that $\Gamma' = F_l * \Delta_1 * \cdots * \Delta_c$, where F_l is a finitely generated Schottky group free on l generators and Δ_i fixes ϵ_i.*

Recall that F_l is a free group with l free generators, which is torsion free. If $P_i = Stab_G(\epsilon_i)$ and P_i^1 is as described above, then $\Delta_i = P_i^1 \cap \Gamma$ and by Theorem 4.1 of [5] it contains a finite index subgroup Δ_i', such that Δ_i' is a cocompact lattice in the unipotent radical U_i of P_i.

We also will use some basic facts from ergodic theory. Given an action of a topological group G on a measure space X with a σ-finite G-quasi-invariant measure μ, we say that the action of G on X is *ergodic*, if for any measurable set $Y \subset X$ such that $\mu(Y \triangle gY) = 0$ for all $g \in G$, we have either $\mu(Y) = 0$ or $\mu(X \setminus Y) = 0$. An automorphism s of the space X is *ergodic* if the action of the group $\{s^m \mid m \in \mathbb{Z}\}$ on X is ergodic.

Let G be an almost simple algebraic group defined over a local field k and $\mathbf{G}(k)^+ = G^+ \subset G$ be a subgroup generated by the elements contained in the unipotent radicals of all minimal parabolic k-subgroups of $\mathbf{G}$. Let $\mathbf{S}$ be a maximal k-split torus of $\mathbf{G}$. Corollary 6.8 of [1] says that there is an integer m, so that $s^m \in G^+$ for any $s \in S$. If G' denotes the closure of $\Gamma \cdot G^+$ in G in the topology induced from that of k, then by Theorem II.7.2 in [4] we can find an $s \in S \cap G^+$ such that s acts ergodically on G'/Γ. It is enough to pick an s whose eigenvalue in a matrix representation of G has the v-value not equal to 1 and take s^m.

3. Auxiliary results

In this section we collect several lemmas and a proposition needed for the proof of the main theorems. Let **G** be an absolutely almost simple algebraic group defined over k of $\mathit{rank}_k\, G = 1$ and $\Gamma \subset G$ be a a non-uniform lattice of infinite index in $\mathit{Comm}_G\, \Gamma$. By Theorem 3 Γ contains a finite index subgroup $\Gamma' = F_l * \Delta_1 * \cdots * \Delta_c$. If $\Gamma' \subset GL_n(\mathcal{A}')$, where $\mathcal{A}'$ is a finitely generated subring of k, then $\Gamma \subset GL_n(\mathcal{A})$, where $\mathcal{A} \supset \mathcal{A}'$ and is also finitely generated. Therefore we can and will assume that $\Gamma = F_l * \Delta_1 * \cdots * \Delta_c$.

Lemma 1. *Let $w \in \Gamma$ be such that $w^m = e$, for some $m \in \mathbb{N}$, then $w \in \gamma \Delta_i \gamma^{-1}$, for some $i \in \{1, 2, \ldots, c\}$, $\gamma \in \Gamma$.*

Proof. By Proposition 2 in [6] w is contained in a conjugate of one of the groups F_l, Δ_i, $i = 1, 2, \ldots, c$. Since F_l is torsion free, the result follows. □

Lemma 2. *Let $e \neq \delta \in \gamma \Delta_j \gamma^{-1}$ for some $\gamma \in \Gamma$, $j \in \{1, 2, \ldots, c\}$. Then the centralizer $C_\Gamma(\delta)$ of δ in Γ is contained in $\gamma \Delta_j \gamma^{-1}$.*

Proof. Conjugating by γ^{-1}, we can assume $\delta \in \Delta_j$. We then want to show that $C_\Gamma(\delta) \subset \Delta_j$.

Let $\gamma \in C_\Gamma(\delta)$ and $\gamma = c_1 c_2 \ldots c_n$ be a reduced word in elements of F_l and Δ_i, $i = 1, 2, \ldots, n$. Then

$$[\gamma, \delta] = c_1 \ldots c_n \delta c_n^{-1} \ldots c_1^{-1} \delta^{-1} = e.$$

If $n > 1$ or $c_n \notin \Delta_j$, we will obtain that a non-trivial word in elements of F_l, Δ_i, $i = 1, 2, \ldots, n$, is equal to e, which contradicts Γ being a free product of F_l and Δ_i, $i = 1, \ldots, c$. Thus $\gamma \in \Delta_j$ and the lemma is proved. □

Corollary 1. *For any at most 2-step nilpotent subgroup $N \subset \Gamma$ such that the center $\mathcal{Z}(N)$ and $N/\mathcal{Z}(N)$ are vector groups over $\mathbb{F}_p$ ($N = \mathcal{Z}(N)$ if N is commutative), there exist $\gamma \in \Gamma$ and $j \in \{1, 2, \ldots, c\}$, such that $N \subset \gamma \Delta_j \gamma^{-1}$.*

Proof. Let $e \neq x \in \mathcal{Z}(N)$, then by Lemma 1 there exist $\gamma \in \Gamma$ and $i \in \{1, 2, \ldots, c\}$ such that $x \in \gamma \Delta_i \gamma^{-1}$. Since $N \subset C_\Gamma(x)$, $N \subset \gamma \Delta_i \gamma^{-1}$ by Lemma 2. □

Recall that $\Delta_i' = \Delta_i \cap U_i$ is a finite index subgroup of Δ_i, which is also a cocompact lattice in the unipotent radical U_i of the minimal parabolic subgroup P_i stabilizing the end ϵ_i.

Lemma 3. *If the intersection $\gamma\Delta_i\gamma^{-1}\cap\Delta_i$ is of finite index in both $\gamma\Delta_i\gamma^{-1}$ and Δ_i, then $\gamma\in P_i$.*

Proof. The condition $\gamma\Delta_i\gamma^{-1}\cap\Delta_i$ is of finite index in both groups implies that $\gamma\Delta_i'\gamma^{-1}\cap\Delta_i'$ is of finite index in $\gamma\Delta_i'\gamma^{-1}$ and Δ_i'. The cocompact lattice Δ_i' is a Zariski dense subgroup of U_i. We can extend $Int\,\gamma$ from Δ_i' to U_i. It is a rational isomorphism of groups, $Int\,\gamma:\mathbf{U}_i\to\mathbf{H}_i$, where $\mathbf{H}_i$ is a closed subgroup of $\mathbf{G}$. Since $Int\,\gamma:\widetilde{\Delta_i}\to\overline{\Delta_i}$, where $\widetilde{\Delta_i}$ and $\overline{\Delta_i}$ are finite index subgroups of Δ_i' and hence Zariski dense in $\mathbf{U}_i$, we conclude that $\mathbf{U}_i=\mathbf{H}_i$. Thus γ normalizes U_i, and is contained in $P_i=N_G(U_i)$. Hence lemma. □

Lemma 4. *There exists a semisimple element $x\in Comm_G(\Gamma)\cap G^+$, $x\notin\Gamma$, such that x acts ergodically on $\Gamma\setminus G'$, where $G'=\overline{\Gamma\cdot G^+}$.*

Proof. Since Γ is of infinite index in $Comm_G\,\Gamma$, we know that $Comm_G\,\Gamma$ is not discrete. By Theorem II.5.1 of [4] the closure of $Comm_G\,\Gamma$ in G in the topology induced from that of k contains G^+. By Proposition 6.14 of [1] G^+ is a closed normal subgroup of G. For any element $g\in G$ let $\lambda(g)$ denote an eigenvalue of g (in the chosen matrix representation) with the maximal v-value. The set Ω of all semisimple elements $x\in G$ with $|\lambda(x)|_v\neq 1$ is open in the topology induced from that of k. Then the set $\Omega\cap G^+$ is open in G^+ and so is $\Omega\cap G^+\setminus\Gamma$. Since $Comm_G\,\Gamma$ is dense in G^+, we can find $x\in Comm_G\,\Gamma$ such that $x\in\Omega\cap G^+\setminus\Gamma$. By Theorem II.7.2 of [4] x acts ergodically on $\Gamma\setminus G'$ and the lemma is proved. □

Let $\mathbf{P}$ be a minimal parabolic subgroup of $\mathbf{G}$. Consider the action of elements of P on $U=R_u(P)$ by conjugation. In particular we are interested in the action of elements of an anisotropic torus $A\subset P$. The torus A acts on $\mathcal{Z}(U)$ by a "rotation" $\phi:A\to GL(\mathcal{Z}(U))$. Since conjugation by elements of P stabilizes $\mathcal{Z}(U)$, we also have an induced action on $U/\mathcal{Z}(U)$. We write $\phi':A\to GL(U/\mathcal{Z}(U))$.

Proposition 1. *Suppose there exists $x\in G$ and $i\in\{1,2,\ldots,c\}$ such that $x\Delta_ix^{-1}\cap\Delta_i$ is of finite index in both groups and*

$$|\alpha_i(x)|_v>C=\max\Big\{\max_{a\in A}\big\{\|\phi(a)\|,\|\phi'(a)\|\big\},p\Big\}$$

(*here we are assuming that the v-value of a uniformizer of k is p^{-1}*). *Then there exists a ring $\mathcal{A}_i$ finitely generated over $\mathbb{F}_p$, so that $\Delta_i\subset GL_n(\mathcal{A}_i)$.*

Proof. Recall $\Delta_i'=\Delta_i\cap U_i$ is a cocompact lattice in U_i and also a finite index subgroup in Δ_i. Note that $x\Delta_ix^{-1}\cap\Delta_i$ is of finite index in both groups implies $x\Delta_i'x^{-1}\cap\Delta_i'$ is of finite index in both groups. Since Δ_i' is of finite index in Δ_i, it is enough to prove the proposition for Δ_i'. To simplify the notation, for the rest of this proposition we will omit subscripts and superscripts, i.e., we will denote by P a minimal parabolic subgroup of $\mathbf{G}$ stabilizing an end ϵ, $U=R_u(P)$ is at most 2-step nilpotent subgroup with $\mathcal{Z}(U)$

and $U/\mathcal{Z}(U)$ being vector groups, $\Delta \subset U$ is a cocompact lattice and $x \in G$ is such that $x\Delta x^{-1} \cap \Delta$ is of finite index in both groups and $|\alpha(x)|_v > C$.

Since $x\Delta x^{-1} \cap \Delta$ is of finite index in both groups we have $x\mathcal{Z}(\Delta)x^{-1} \cap \mathcal{Z}(\Delta)$ is of finite index in both groups. By Lemma 3 $x \in P$ and $x = s \cdot a \cdot u$, where $s \in S$, a maximal k-split torus, $a \in A$, an anisotropic torus, and $u \in U = R_u(P)$. Therefore, for $z \in \mathcal{Z}(\Delta)$

$$xzx^{-1} = \alpha(s)\phi(a)z.$$

Let Z' denote $x\mathcal{Z}(\Delta)x^{-1} \cap \mathcal{Z}(\Delta)$ and $V = \mathcal{Z}(\Delta)/Z'$. Since $\mathcal{Z}(\Delta)$ is a vector space over $\mathbb{F}_p$, then V is a finite dimensional vector space over $\mathbb{F}_p$ and $\mathcal{Z}(\Delta) = V \oplus Z'$. For any $u \in \mathcal{Z}(\Delta)$ we have $u = v_1 + b_1$, where $v_1 \in V$ and $b_1 \in Z'$. Thus $b_1 = \alpha(s)^2\phi(a)u_2$, where $u_2 \in \mathcal{Z}(\Delta)$. Note that if $|\alpha(s)|_v > C$, then $\|u\| \geqslant \|b_1\| > \|u_2\|$. We continue and write $u_2 = v_2 + b_2$, with $v_2 \in V$ and $b_2 \in Z'$. Then $b_2 = \alpha(s)^2\phi(a)u_3$ with $u_3 \in \mathcal{Z}(\Delta)$ and $\|u_2\| \geqslant \|b_2\| > \|u_3\|$. If we continue this process, we must stop in a finite number of steps, since $\|u_j\|$ are decreasing and $\mathcal{Z}(\Delta)$ being a lattice in $\mathcal{Z}(U)$ does not contain elements with norm smaller then a certain value.

For any $v \in V$ let v_{ij} denote the coordinates of v in the selected matrix representation of G. Since V is a finite dimensional vector space over $\mathbb{F}_p$, the set $\{v_{ij} \mid v \in V\}$ is finite. Let ϕ_{ij} denote the coordinates of $\phi(a)$ and let $\mathcal{A}_1 = \mathbb{F}_p[v_{ij}, \phi_{ij}, \alpha(s)]$ a finitely generated ring. Then $\mathcal{Z}(\Delta) \subset GL_n(\mathcal{A}_1)$.

Consider the canonical projection $pr : U \to U/\mathcal{Z}(U)$ and denote by $\bar{\cdot}$ the images in $U/\mathcal{Z}(U)$. Then $\overline{x\Delta x^{-1}} \cap \overline{\Delta} = \alpha(s)\phi'(a)\overline{\Delta} \cap \overline{\Delta} = \overline{\Delta}'$ is of finite index in both groups. Write $\overline{\Delta} = \overline{\Delta}' \oplus \overline{W}$, where $\overline{W}$ is a finite dimensional vector space over $\mathbb{F}_p$. For any $\bar{w} \in \overline{W}$, choose $w \in pr^{-1}(\overline{W})$, and let w_{ij} be the coordinates of w in the chosen matrix representation of G. Then arguing as in the case of $\mathcal{Z}(\Delta)$ one can see that $\Delta \subset GL_n(\mathcal{A})$, where $\mathcal{A} = \mathbb{F}_p[\alpha(s), \phi_{ij}, \phi'_{ij}, v_{ij}, w_{ij}]$ is a finitely generated ring. □

It is easy to show that the following is true.

Lemma 5. *Suppose $\Delta_i \subset GL_n(\mathcal{A}_i)$, where $\mathcal{A}_i$ is a finitely generated ring, and $x\Delta_i x^{-1} \cap \Delta_j$ is of finite index in both groups, for some $x \in G$. Then there exists a ring $\mathcal{A}_j$ finitely generated over $\mathbb{F}_p$, such that $\Delta_j \subset GL_n(\mathcal{A}_j)$.*

4. Proof of the theorems

We now can prove our main theorem.

Theorem 2. *Let $k \cong \mathbb{F}_q((1/t))$, **G** is a k-rank 1 absolutely almost simple algebraic group defined over k and Γ is a non-uniform lattice in G that has an infinite index in its commensurator $\mathrm{Comm}_G\,\Gamma$. Then there exists a ring $\mathcal{A}$ finitely generated over $\mathbb{F}_p$, such that $\Gamma \subset GL_n(\mathcal{A})$.*

Proof. Fix a matrix representation of $G \subset GL_n(k)$. By Theorem 7.1 in [2] (see Section 2) we can assume $\Gamma = F_l * \Delta_1 * \Delta_2 * \cdots * \Delta_c$, where F_l is a Schottky group on l generators and $\Delta_i \subset P_i$, where $P_i = Stab_G\, \epsilon_i$ stabilizes the ends ϵ_i of the Bruhat–Tits tree X of G.

Let $x \in Comm_G\, \Gamma \setminus \Gamma$ be as in Lemma 4. Consider the set of subgroups $x^m \Delta_1 x^{-m}$ of G as m runs through the set of integers $\mathbb{N}$. By Lemma 1 there exist $\gamma_m \in \Gamma$ and $i_m \in \{1, 2, \ldots, c\}$ such that

$$x^m \Delta_1 x^{-m} \cap \Gamma \subset \gamma_m \Delta_{i_m} \gamma_m^{-1}.$$

In fact, it is easy to see that

$$x^m \Delta_1 x^{-m} \cap \gamma_m \Delta_{i_m} \gamma_m^{-1}$$

must be of finite index in both groups. Let $I' \subset \{1, 2, \ldots, c\}$ be a subset such that any $i \in I'$ is repeated infinitely many times as an index i_m, $m \in \mathbb{N}$, then $|I'| \geqslant 1$. Let M_0 be the smallest such that $i_{M_0} \in I'$, and M_1 be the smallest such that $I' \subset \{i_{M_0}, \ldots, i_{M_1}\}$. Let $M = M_1 - M_0$. Let $m(j)$ be the smallest integer such that $i_{m(j)} = j$ for $j \in I'$, note that $M_0 \leqslant m(j) \leqslant M_1$. Without the loss of generality we can and will assume $I' = \{1, 2, \ldots, c\}$ and $M_0 = 1$.

Consider the subgroups $\gamma_{m(j)} \Delta_j \gamma_{m(j)}^{-1}$, $j = 1, 2, \ldots, c$, and let $D_\Sigma = \max_{j \in I'} \Sigma(\gamma_{m(j)} \times \Delta_j \gamma_{m(j)}^{-1})$ (see Section 2 for the definition of $\Sigma(\Delta)$). For the minimal parabolic subgroups $P_j = Stab_G\, \epsilon_j \supset \Delta_j$, let $g_j \in G$ be such that $g_j \gamma_{m(j)} P_j \gamma_{m(j)}^{-1} g_j^{-1}$ is in the block upper triangular form in the chosen matrix representation of G. Let $R = \max_{j \in I'} \|g_j\|, \|g_j^{-1}\|$. Fix a maximal torus $\mathbf{T}_j$ of $\mathbf{P}_j$, then $T_j = S_j \cdot A_j$, where $\mathbf{S}$ is a maximal k-split torus and $\mathbf{A}_j$ is an anisotropic torus. Let $D_A = \max_{a \in A_j, j \in I'} \|\gamma_{m(j)} A_j \gamma_{m(j)}^{-1}\|$, it exists since each A_j is compact. Let $C = \max_{j \in I'} C_j$, where C_j are constants chosen as in Proposition 1 for the groups $\gamma_{m(j)} \Delta_j \gamma_{m(j)}^{-1}$. Let $d = D_\Sigma D_A R^2 (C + 1)$. Let $G(d) = \{g \in G \subset GL_n(k) \mid \|g\| \leqslant d\}$, and $K(d)$ be the canonical projection of $G(d) \cap G'$ on $\Gamma \setminus G'$. For $g \in G$, let $\bar{g}$ denote the projection of g. By ergodicity and continuity of the action of x on $\Gamma \setminus G'$ there exists and $N = N(d, M) \in \mathbb{N}$ such that $\bar{x}^{N-M}, \bar{x}^{N-M+1}, \ldots, \bar{x}^N \notin K(d)$. Then one can find $j \in I'$ and $r \in \{N - M, \ldots, N - 1, N\}$ such that

$$x^r \gamma_{m(j)} \Delta_j \gamma_{m(j)}^{-1} x^{-r} \cap \gamma_N \Delta_j \gamma_N^{-1}$$

is of finite index in both groups. We can rewrite the above expression as

$$\gamma_{m(j)} \Delta_j \gamma_{m(j)}^{-1} \cap \left(x^{-r} \gamma_N \gamma_{m(j)}^{-1}\right) \gamma_{m(j)} \Delta_j \gamma_{m(j)}^{-1} \left(\gamma_{m(j)} \gamma_N^{-1} x^r\right)$$

is of finite index in both groups. By Lemma 3 $x^{-r} \gamma_N \gamma_{m(j)}^{-1} \in \gamma_{m(j)} P_j \gamma_{m(j)}^{-1}$ and thus

$$x^{-r} \gamma_N \gamma_{m(j)}^{-1} = s \cdot a \cdot u, \tag{1}$$

where $s \in \gamma_{m(j)} S_j \gamma_{m(j)}^{-1}$, $a \in \gamma_{m(j)} A_j \gamma_{m(j)}^{-1}$ and $u \in \gamma_{m(j)} U_j \gamma_{m(j)}^{-1}$.

We can assume that $u \in G(D_\Sigma)$, otherwise we can multiply Eq. (1) by an element of $\gamma_{m(j)}\Delta_j\gamma_{m(j)}^{-1}$ on the right. Also recall that $a \in G(D_A)$ by definition of D_A. Since $x^{-r}\gamma_N\gamma_{m(j)}^{-1} \notin G(d)$, we can conclude that $s \notin G(d/D_\Sigma D_A)$ or $\|s\| \geqslant d/(D_\Sigma D_A)$. Since $s' = g_j s g_j^{-1}$ is a diagonal element, $\|s'\| = |\lambda(s')|_v$. Then $s = g_j^{-1} s' g_j$ and $\|s'\| \geqslant \|s\|/R^2$. We can conclude that

$$\left|\lambda(s)\right|_v = \left|\lambda(s')\right|_v \geqslant \frac{d}{D_\Sigma D_A R^2} > C.$$

Since for any $s \in S$ s acts on $\gamma_{m(j)}\Delta_j\gamma_{m(j)}^{-1}$ by multiplication by $\lambda^2(s)$ on the center and by $\lambda(s)$ on the factor group by the center, we can apply Proposition 1 to the subgroup $\gamma_{m(j)}\Delta_j\gamma_{m(j)}^{-1}$ and conclude that there exists a ring $\mathcal{A}'_j$ finitely generated over $\mathbb{F}_p$ such that $\gamma_{m(j)}\Delta_j\gamma_{m(j)}^{-1} \subset GL_n(\mathcal{A}'_j)$. Let $\mathcal{A}_j$ be a ring obtained from $\mathcal{A}'_j$ by adjoining the coordinates of $\gamma_{m(j)}$, then $\Delta_j \subset GL_n(\mathcal{A}_j)$.

Note also that by Lemma 5 there exists a finitely generated ring $\mathcal{A}_1$, such that $\Delta_1 \subset GL_n(\mathcal{A}_1)$, and, more generally, for each i_m with $m \leqslant N(d, M)$ there exists a finitely generated ring $\mathcal{A}_{i_m}$ such that $\Delta_{i_m} \subset GL_n(\mathcal{A}_{i_m})$.

Let $J = \{1, i_1, \ldots, i_m\}$ for $m \leqslant N(d, M)$. If $J = \{1, 2, \ldots, c\}$, then take $\mathcal{A}$ to be a ring generated by the generators of all $\mathcal{A}_j$, $j \in J$, and the coordinates of the l generators of F_l. Then $\mathcal{A}$ is finitely generated over $\mathbb{F}_p$ and $\Gamma \subset GL_n(\mathcal{A})$.

Now suppose $J \neq \{1, \ldots, c\}$. Then choose $r \in \{1, 2, \ldots, c\} \setminus J$ and consider the conjugates $x^m\Delta_r x^{-m}$ as m runs through $\mathbb{N}$. Repeating the previous argument, we will be able to conclude that $\Delta_r \subset GL_n(\mathcal{A}_r)$, for some finitely generated ring $\mathcal{A}_r$. By possibly repeating this argument finite number of times, we will conclude that $\Delta_i \subset GL_n(\mathcal{A}_i)$, $\mathcal{A}_i$ finitely generated over $\mathbb{F}_p$, for all $i \in \{1, 2, \ldots, c\}$. Then the conclusion of the theorem will follow. □

We now can derive the arithmeticity of the lattice Γ.

Theorem 1. *Let* $\mathbf{G}$ *be a simple algebraic group defined over a local field* k *of positive characteristic with* $\operatorname{rank}_k G = 1$. *Let* $\Gamma \subset G$ *be a non-uniform lattice that has an infinite index in its commensurator* $\mathrm{Comm}_G\,\Gamma$. *Then* Γ *is arithmetic.*

Proof. The following proof is basically the proof given by G. Margulis for his arithmeticity theorem. We reproduce it here with some abbreviations for the sake of completeness. It works under the assumption that if characteristic of k is 2, then $\mathbf{G}$ is not of type $\mathbb{A}_1$, $\mathbb{C}_2$ or $\mathbb{C}_3$. The problem with these three groups is that they have non-central isogenies. If $f : G \to G'$ is a surjective k-homomorphism of algebraic groups and $df : \mathit{Lie}\, G \to \mathit{Lie}\, G'$ is the corresponding homomorphism of the Lie algebras, we say that f is *central* if the kernel $\ker df$ is central in $\mathit{Lie}\, G$. Whereas, in general, there are some other groups possessing non-central isogenies, only the groups listed above have forms of rank 1 over local fields of positive characteristic. To obtain the proof in the three exceptional cases one needs to apply the methods developed by T.N. Venkataramana in [7]. We will briefly indicate these applications as remarks in the proof. □

Step 1. Without loss of generality we can assume that $\mathbf{G}$ is an adjoint group. It is because the rank of $\mathbf{G}$, Γ being of infinite index in $Comm_G\,\Gamma$ and arithmeticity of a lattice are preserved by a central isogeny.

Step 2. Let $Ad : \mathbf{G} \to GL(Lie\,\mathbf{G})$ be the adjoint representation. Denote by $L \subset k$ the field generated by the set $Tr\,Ad\,\Gamma$. By IX.1.8 of [4] $Ad\,\Gamma$ is definable over L. Fix the basis in $Lie\,\mathbf{G}$ in which elements of $Ad\,\Gamma$ are written as matrices with entries in L. Since Γ is Zariski dense in $\mathbf{G}$, $Ad\,\Gamma$ is Zariski dense in $Ad\,\mathbf{G}$. But $Ad\,\Gamma \subset GL_m(L)$, so $Ad\,\mathbf{G}$ is an L-subgroup of GL_m. Since $\mathbf{G}$ is adjoint and Ad is central, it is an isomorphism. Thus identifying $\mathbf{G}$ and $Ad\,\mathbf{G}$ we can assume that $\mathbf{G}$ is defined over L and $\Gamma \subset \mathbf{G}(L)$. Also we will assume that through this identification we fixed a matrix representation of G.

Remark. If k is of characteristic 2 and $\mathbf{G}$ is one of the exceptional types listed above, then Proposition IX.1.8 of [4] does not apply. Here we denote by $L' \subset k$ the subfield generated by $\{(Ad(\gamma))_{ij}\}$ and by L the field generated by $Tr\,Ad\,\Gamma$ (note $L \subset L'$) and conclude that $\mathbf{G}$ is defined over L' and $\Gamma \subset \mathbf{G}(L')$ as above.

Step 3. Let l be a local field and $\sigma : L \to l$ a ring homomorphism with $\sigma(L)$ dense in l. It induces a group homomorphism $\sigma^* : G \to^{\sigma} G$.

Lemma 6. *With the above notation either $\sigma^*(\Gamma)$ is relatively compact in ${}^{\sigma}G(l)$ or there exists a* continuous *isomorphism $\theta : k \to l$ such that $\sigma^* = \Psi \circ \theta^*$, where Ψ is a rational k-isomorphism.*

Proof. By Step 2 $\Gamma \subset G(L)$ hence σ^* is defined on Γ. By Lemma VII.6.2 of [4] $Comm_G\,\Gamma \subset G(L)$, hence σ^* is defined on $Comm_G\,\Gamma$. Then apply the superrigidity theorem [4, Theorem VII.5.4] to obtain a continuous embedding $\theta : k \to l$ and a rational k-isogeny Ψ. Since $\mathbf{G}$ is adjoint, Ψ is a k-isomorphism. Then $\sigma(Tr(\gamma)) = \theta(Tr(\gamma))$ and $\theta(k) \supset \sigma(L)$. But $\sigma(L)$ is dense in l. Thus the continuous homomorphism $\theta : k \to l$ is an isomorphism. □

From this we can draw the following corollary.

Corollary 2. $\sigma(Tr(\gamma)) = \theta(Tr(\gamma))$, *for all* $\gamma \in \Gamma$.

If $g \in GL_m(k)$ is such that $\{g^n \mid n \in \mathbb{N}\}$ is relatively compact in $GL_m(k)$ then the absolute value of any eigenvalue of g is 1 and $|\,Tr(g)| \leqslant m$. Hence

Lemma 7. *For any $\gamma \in \Gamma$ set $c(\gamma) = \max\{\dim Lie\,\mathbf{G}, |\,Tr(\gamma)|\}$. Then $|\sigma(Tr(\gamma))| \leqslant c(\gamma)$.*

Remark. If $\mathbf{G}$ is one of the exceptions listed at the beginning of the theorem, then one considers $\sigma : L' \to l$. Whereas there is still a continuous field homomorphism $\theta : k \to l$ such that $\sigma^* = \Psi \circ \theta^*$, we cannot conclude that Ψ is an isomorphism, since it can be a non-central isogeny. In this case the connection between $\sigma(Tr(\gamma))$ and $\theta(Tr(\gamma))$ is given in Lemma 5.9 of [7].

Step 4. L is a global field.

Recall $\Gamma \subset \mathbf{G}(L)$ by Step 2. Denote by L' the field generated by the entries of matrices in Γ. Obviously $L' \subset L$. Since $Tr(\Gamma) \subset L'$ we also have that $L \subset L'$ and hence $L = L'$. By Theorem 2 L' is finitely generated.

Suppose L is not a global field. Then the transcendence degree of L over its prime subfield $\deg_{tr} L \geqslant 2$. Since $Tr(\Gamma)$ generates L and L is finitely generated, there exists $\gamma \in \Gamma$ such that $Tr(\gamma)$ is transcendental over the prime subfield of L. By Lemma IX.2.9 in [4] there exists a local field l and a homomorphism $\sigma : L \to l$ such that $\sigma(L)$ is dense in l and $|\sigma(Tr(\gamma))|_l \geqslant c(\gamma)$. This contradicts Lemma 7 in Step 3. Hence L is global.

Remark. From Step 3 we can only conclude that $\sigma^* = \Psi \circ \theta^*$. It follows then, that

$$\sigma\big(Tr(\gamma)\big) = \theta\big(Tr\big(\theta^{-1}\Psi\theta\big)(\gamma)\big),$$

where $\theta^{-1}\Psi\theta$ is an isogeny (possibly non-central). One uses Lemmas 5.15 and 5.16 of [7] to conclude that L is global. Proposition 5.22 [7] shows that L' is global as well and can be taken to be the inseparable closure of L.

Step 5. Γ is arithmetic.

By Step 4 $\Gamma \subset \mathbf{G}(L)$, where L is a global field and $L \subset k$, where k is a local field. Assume k is the completion of L with respect to the valuation v. Let $w = v^{-1}$ be a valuation on L defined by $|l|_w = 1/|l|_v$. Let L_w denote the completion of L with respect to w and $\mathcal{O}_w$ denote the ring of integers of L_w. Since $\mathbf{G}(\mathcal{O}_w)$ is open in $\mathbf{G}(L_w)$, the subgroup $\Gamma \cap \mathbf{G}(\mathcal{O}_w)$ is of finite index in Γ and, hence, is also a lattice in $\mathbf{G}(k)$. Since $\Gamma \cap \mathbf{G}(\mathcal{O}_w) \subset \mathbf{G}(\mathcal{O}_w)$ and $\mathbf{G}(\mathcal{O}_w)$ is discrete in $\mathbf{G}(k)$, we conclude that $\Gamma \cap \mathbf{G}(\mathcal{O}_w)$ is of finite index in $\mathbf{G}(\mathcal{O}_w)$. Hence Γ is commensurable with $\mathbf{G}(L(S))$, with $S = \{v\}$.

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